

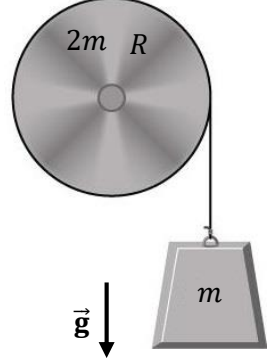


PHYS 101 – General Physics I Final Exam

Duration: 120 minutes

Tuesday, 21 May 2024; 09:30

1. A solid cylinder of mass $2m$ and radius R is mounted on frictionless bearings. A cord is wrapped around the rim of the cylinder and a mass m is fixed at the end of the cord. The cord does not slip on the cylinder. When released, the mass m accelerates down under the action of constant gravitational acceleration g . Moment of inertia for the cylinder is $I = mR^2$.



(a) (8 Pts.) Use the work energy theorem to calculate the angular velocity of the cylinder after the mass falls down a vertical distance h .

Assume now that there is friction at the bearing, and the speed of the mass m after falling a distance h is $\sqrt{(gh/2)}$.

(b) (8 Pts.) Find the tension in the cord.

(c) (9 Pts.) Find the expression for the friction torque.

Solution: (a) Let v denote the speed of the mass after it falls a distance h . Since the cord does not slip on the cylinder, we have $v = R\omega$, where ω is the angular speed of the cylinder. Total energy is conserved during the motion, so

$$E_i = mgh, \quad E_f = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mR^2\omega^2 + \frac{1}{2}(mR^2)\omega^2 = mR^2\omega^2, \quad E_f = E_i \rightarrow \omega = \frac{1}{R}\sqrt{gh}.$$

(b) In this part there is frictional torque. The downward acceleration a of the mass can be found as

$$v_f^2 = 2ah \rightarrow \frac{gh}{2} = 2ah \rightarrow a = \frac{g}{4}.$$

Newton's second law applied to the mass gives

$$mg - T = ma \rightarrow T = mg - m\frac{g}{4} \rightarrow T = \frac{3}{4}mg.$$

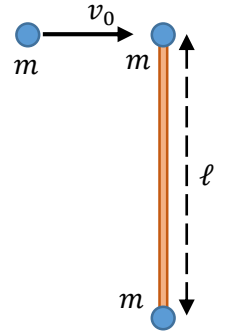
(c) Angular acceleration of the cylinder can be found as

$$\alpha = \frac{a}{R} = \frac{g}{4R}.$$

This means

$$RT - \tau_f = I\alpha \rightarrow \frac{3}{4}mgR - \tau_f = (mR^2)\left(\frac{g}{4R}\right) \rightarrow \tau_f = \frac{1}{2}mgR.$$

2. A massless rigid rod of length ℓ has two small objects each of mass m fixed at the two ends, and is at rest on a frictionless horizontal surface. The figure shows the view of the situation as seen from the top. Another small object with mass m sliding on the horizontal surface with speed v_0 collides with the top object perpendicular to the rod as shown. The two colliding objects stick to each other after the collision.



- (a) (6 Pts.) Immediately after the collision what is the distance of the center of mass of the system from the top mass?
- (b) (6 Pts.) What is the speed of the center of mass of the system after the collision?
- (c) (7 Pts.) What is the angular speed of the system about its center of mass after the collision?
- (d) (6 Pts.) What ratio of the initial energy is lost in the collision?

Solution: (a) Measured from the top mass, position of the center of mass immediately after the collision is

$$y_{CM} = \frac{m\ell}{3m} = \frac{1}{3}\ell.$$

(b) Linear momentum is conserved in the collision. If V_{CM} denotes the velocity of the center of mass after the collision,

$$mv_0 = 3m V_{CM} \rightarrow V_{CM} = \frac{1}{3}v_0.$$

(c) Angular momentum with respect to the center of mass is also conserved. Letting ω denote the angular speed of the system after the collision,

$$mv_0 \frac{\ell}{3} = I\omega, \quad I = 2m\left(\frac{\ell}{3}\right)^2 + m\left(\frac{2\ell}{3}\right)^2 = \frac{2}{3}m\ell^2 \rightarrow \frac{1}{3}mv_0\ell = \frac{2}{3}m\ell^2\omega \rightarrow \omega = \frac{v_0}{2\ell}.$$

(d)

$$E_i = \frac{1}{2}mv_0^2, \quad E_f = \frac{1}{2}(3m)V_{CM}^2 + \frac{1}{2}I\omega^2 = \frac{1}{4}mv_0^2.$$

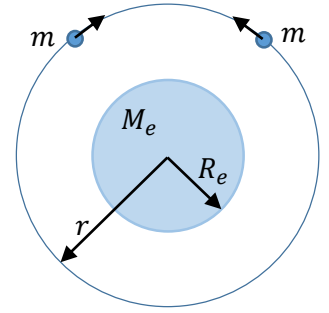
This means half the initial energy is lost, i.e., $\Delta E/E_i = -1/2$.

3. Consider two small satellites A and B of equal mass m , moving in the same circular orbit of radius r around the earth E (mass $M_e \gg m$) but in opposite senses of rotation and therefore on a collision course, as illustrated in the figure. (Ignore the gravitational attraction between the satellites.)

(a) (9 Pts.) In terms of G, M_e, m and r , find the total mechanical energy of the two satellites before the collision.

(b) (8 Pts.) If the collision is completely inelastic so that wreckage remains as one piece of tangled material (mass $2m$), find the total mechanical energy immediately after the collision.

(c) (8 Pts.) What will be the speed of the wreckage just before it hits the surface of the earth (radius R_e)?



Solution: Both satellites are in circular orbit. The relation between the radius of the orbit and the speed of the satellites is

$$G \frac{mM_e}{r^2} = m \frac{v^2}{r} \rightarrow v^2 = \frac{GM_e}{r}.$$

(a) Total mechanical energy of each satellite is

$$E = \frac{1}{2}mv^2 - G \frac{mM_e}{r} = -G \frac{mM_e}{2r}.$$

Therefore, for two satellites,

$$E = -G \frac{mM_e}{r}.$$

(b) Just before the collision, total momentum of the two satellites is zero. Therefore after the collision speed of the wreckage will be zero. Therefore, the total mechanical energy immediately after the collision is

$$E = -G \frac{(2m)M_e}{r} = -2G \frac{mM_e}{r}.$$

(c) Energy will be conserved during the fall. So if v denotes the speed of the wreckage at the instant it hits the surface of the earth, we have

$$E_i = -2G \frac{mM_e}{r}, \quad E_f = \frac{1}{2}(2m)v^2 - 2G \frac{mM_e}{R_e},$$

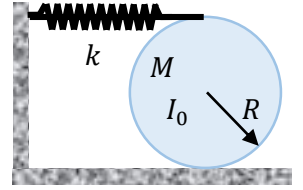
and

$$E_i = E_f \rightarrow v = \sqrt{2GM_e \left(\frac{1}{R_e} - \frac{1}{r} \right)} = \sqrt{\frac{2GM_e}{rR_e}} (r - R_e).$$

4. A solid cylinder of mass M and radius R has a moment of inertia I_0 around its center. The cylinder is placed on a horizontal surface with friction, and is connected to a vertical wall with a spring with stiffness constant k from its top, as shown in the figure. The cylinder is displaced slightly from the equilibrium point stretching the spring by a small amount and is released causing it to roll without slipping on the horizontal surface.

(a) (5 Pts.) Draw a free body diagram.

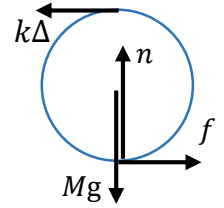
(b) (20 Pts.) Find the frequency of small oscillations of the cylinder.



Solution:

If the cylinder rolls through a small angle θ ($\sin \theta \approx \theta$) to the right, the spring will be stretched by Δ , where

$$\Delta = 2R\theta.$$



Considering the horizontal forces shown in the free body diagram, evaluating the torque with respect to the point of contact, and using the rolling condition $a = R\alpha$ we write,

$$\tau = 2Rk\Delta, \quad \tau = (I_0 + MR^2)\alpha \rightarrow \alpha = \frac{2kR\Delta}{I_0 + MR^2} = \left(\frac{4kR^2}{I_0 + MR^2} \right) \theta.$$

The angular acceleration is in the decreasing θ direction, so we have

$$\alpha = -\frac{d^2\theta}{dt^2} \rightarrow \frac{d^2\theta}{dt^2} = -\left(\frac{4kR^2}{I_0 + MR^2} \right) \theta \rightarrow \omega = \sqrt{\frac{4kR^2}{I_0 + MR^2}}.$$

Alternate method: Total energy of the system is

$$E = \frac{1}{2}(I_0 + MR^2)\omega^2 + \frac{1}{2}k\Delta^2 \rightarrow E = \frac{1}{2}(I_0 + MR^2)\omega^2 + 2kR^2\theta^2$$

Since the total energy is constant in time

$$\frac{dE}{dt} = \omega \left[(I_0 + MR^2) \frac{d^2\theta}{dt^2} + 4kR^2\theta \right] = 0 \rightarrow \frac{d^2\theta}{dt^2} = -\left(\frac{4kR^2}{I_0 + MR^2} \right) \theta.$$